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THE INCLINED ENTRY OF A THIN WEDGE INTO AN INCOMPRESSIBLE FLUID†

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The self-similar problem of the inclined entry of a thin wedge into a half-space filled with an ideal incompressible fluid is considered in a linear formulation. The different modes of fluid motion whose existence has been previously demonstrated [1] are investigated. A criterion for non-separated flow is obtained in the form of a relation between three angles, defining the angle of the wedge, the direction of the entry velocity and the angle of attack. If this relation is not satisfied, modes of motion are possible in which a cavity is adjacent to one of the faces of the wedge. If the pressure in the cavity is less than the pressure at the surface of the fluid half-space, then only two of these modes exist and both faces are always wetted by the fluid, even in the case when the angle of the wedge is zero. If the pressure in the cavity is equal to the pressure at the surface of the half-space, another mode of motion exists: one of the faces of the wedge is not wetted by the fluid. A criterion is obtained for the transition from this mode to the mode with a cavity. The dependence of the size of the cavity and the force acting on the faces of the wedge on the parameters of the problem is investigated numerically.

1. We consider the self-similar problem of the entry of a thin rigid wedge at constant velocity $\mathbf{U}$ into an ideal incompressible weightless fluid half-space $Y \leqslant 0$, $-\infty < X < \infty$. The simplest linear approximation [2] is used, which enables us to carry out the investigation for all parameter values admissible for a thin wedge. It has been shown [1] that there are three types of fluid motion. One of them is non-separated flow. The other two are separated, the streams separating at the edge of the wedge C (Fig. 1). One of the faces of the wedge may turn out to be unwetted by the fluid. This is the case when a plate enters the fluid. The third type is associated with the formation of a cavity CD on one of the faces. To fix our ideas, we shall assume that the cavity is attached to the left face CB. Three angles β, α_1 and α_2 define the direction of the velocity $\mathbf{U}$, the spatial orientation and the angle $\alpha=\alpha_1+\alpha_2$ of the wedge (Fig. 1). For the case of the fluid motion with a cavity there is yet another parameter p_0, the difference between the pressure at the free boundary of the half-space and the pressure in the cavity. Which of the three types of fluid motion occurs depends on the relations between these parameters. It is always assumed that $p_0 \geqslant 0$, $\alpha_1 \geqslant 0$. Without loss of generality we can assume the pressure in the cavity to be zero.

It is convenient to introduce dimensionless self-similar coordinates $x=X/(Ut)$, $y=Y/(Ut)$, pressure $p=P/(\rho_0 U^2)$, and mass velocity $\mathbf{v}=\mathbf{V}/U$, where ρ_0 is the density and t is the time, together with the complex coordinate $z=x+iy$ and the complex velocity $V=\upsilon_x-i\upsilon_y$.

The pressure can be written in the form

$$p=\operatorname{Re}P(z)$$
$$P(z)=zV(z)-\int_{-\infty}^{z} V(\tau)d\tau-\frac{1}{2}|V(z)|^2+p_0 \tag{1.1}$$

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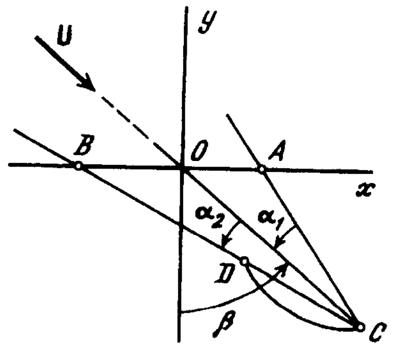


Fig. 1.

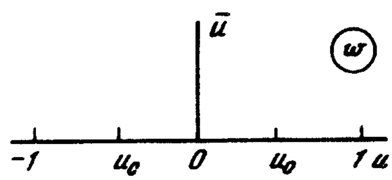


Fig. 2.

By a thin wedge we not only mean that the angle of the wedge α is small, but also that α_1 and α_2 are small. Then the perturbation of the fluid will also be small: $\upsilon \ll 1$. The approximation associated with the smallness of α_1 and α_2 is as follows. The boundary conditions relate to the x axis and the section of the line OC, and the term $1/2|V(z)|^2$ in (1.1) is neglected. At the boundary of the half-space we have $p = p_0$. It follows from (1.1) that this is equivalent to the condition $\upsilon_x(x, 0) = 0$. If we denote the radius-vector of points of the line OC by $\mathbf{r}$, then the pressure along this line is given by

$$p(r) = \upsilon_r r - \int_0^r \upsilon_r(r)dr + p_0 \quad (r = |\mathbf{r}|, \ \upsilon_r = r^{-1}(\mathbf{v}, \mathbf{r})) \tag{1.2}$$

At the boundary of the cavity CD the condition $p = 0$ is satisfied. From this condition and from (1.2) it follows that $\upsilon_r = \upsilon_0 = \text{const}$ on CD. Along the parts AC and BD of the faces of the wedge the impermeability condition must be satisfied.

2. The fluid flow domain can be conformally mapped into the upper half-plane $\operatorname{Im} w > 0$, $w = u + i\bar{u}$. This mapping has the form

$$z = z_c f(w)/f(u_c)$$

$$u_c = -2\delta, \ \delta = \beta/\pi, \ -\tfrac{1}{2} < \delta < \tfrac{1}{2}, \ z_c = -ie^{i\pi\delta}$$

$$f(w) = (1-w)^{\frac{1}{2}+\delta}(1+w)^{\frac{1}{2}-\delta}, \ \arg f(0) = 0$$

Cuts for the function $f(w)$ are chosen in the lower half-plane $\operatorname{Im} w < 0$. The points A, C, B and D become points on the real axis $\operatorname{Im} w = 0$: $u = -1$, $u = u_C$, $u = u_0$, $u = 1$, respectively (Fig. 2).

The size of the cavity is given by the formula

$$l(u_0) = |z(u_c) - z(u_0)| = 1 - f(u_0)/f(u_c)$$

$$u_c \leqslant u_0 \leqslant 1, \ 0 \leqslant l \leqslant 1, \ l(1) = 1, \ l(u_c) = 0$$

Thus the problem under consideration reduces to finding a function $V(w)$, analytic in the upper half-plant $\operatorname{Im} w > 0$, and satisfying the conditions

$$\operatorname{Re} V = 0, \ |u| > 1; \ \operatorname{Re}(z_c V) = \upsilon_0, \ u_c < u < u_0 \tag{2.1}$$

$$\operatorname{Im}(z_c V) = \begin{cases} -\alpha_1, & -1 < u < u_c \\ \alpha_2, & u_0 < u < 1 \end{cases} \tag{2.2}$$

$$F(u_0) \equiv \upsilon_0 - \int_{-1}^{u_c} \mathrm{Re}[V(u)z'(u)]du = -p_0 \tag{2.3}$$

The first condition of (2.1) means that the pressure along the free boundary $y = 0$ is constant, and the second condition is the condition for the pressure along the free surface of the cavity to be constant. Condition (2.2) is the impermeability condition on the faces of the wedge. Condition (2.3) means that the pressure at the boundary of the cavity is zero. It is also necessary to require the function $V(z)$ to decrease faster than $1/z$ as $z \to \infty$, and then $p(z) \to p_0$ as $z \to \infty$.

A function $V(w)$ satisfying conditions (2.1) and (2.2) is sought in the form

$$V(w) = \varphi(w) \int_{-\infty}^{\infty} \frac{g(u)du}{u - w} \tag{2.4}$$

where $\varphi(w)$ is the solution of the problem under consideration, but with homogeneous conditions, i.e. with conditions (2.1) and (2.2) in which $\upsilon_0 = \alpha_1 = \alpha_2 = 0$. Choosing some function $\varphi(w)$, one can select a function $g(u)$ so that conditions (2.1) and (2.2) are satisfied. It can be shown that the general solution of the problem can be represented in the form of the sum of a completely defined solution of the inhomogeneous equation (in (2.4) we choose a given function $\varphi(w) = \varphi_0(w)$) and the general solution of the homogeneous equation $\varphi(w)$. The latter is a linear combination with real coefficients of the functions

$$i(w-1)^{-\delta+m_1}(w+2)^{\delta+m_2}(w-u_c)^{n_c-1/2}(w-u_0)^{n_0-1/2}$$

Here m_1, m_2, n_c, n_0 are arbitrary integers.

For the complex velocity one obtains

$$V(w) = \frac{\varphi_0(w)}{\pi}[\alpha_2 I(u_0, 1) - \alpha_1 I(-1, u_c) - \upsilon_0 I(u_c, u_0)] + \varphi(w)$$

$$I(x, y) = \int_x^y \frac{\xi(t)dt}{t - w}, \quad \xi(u) = \frac{1}{|\varphi_0(u)|} \tag{2.5}$$

$$\varphi_0(w) = i\left(\frac{w-1}{w+1}\right)^{-\delta}\left(\frac{w-u_0}{w-u_c}\right)^{1/2}, \quad \arg\varphi(u) = \pi/2 \quad \text{for } u > 1$$

From the requirements that $V(w) \to 0$ as $w \to \infty$, and that the energy, fluid energy flux, and the forces acting on the wedge faces are finite, it follows that

$$m_1 = m_2 = n_c = n_0 = 0, \quad \varphi(w) = C\varphi_0(w)/(w - u_0)$$

where C is a real constant.

If one chooses a neighbourhood of the point D in the form of a semicircle of radius R, then $\varphi(w) = O(R^{-1/2})$ as $R \to 0$. It can be shown that the energy flux across the circumference of this semicircle as $R \to 0$ is of order unity and does not depend on R. The energy flux across the other part of the boundary of the semicircle, corresponding to parts of the impermeable boundary and the boundary of the cavity, is of order $R^{1/2}$ and vanishes as $R \to 0$. Hence the function $\varphi(w)$ is associated with an energy source situated at the point D. If $C > 0$, then there is energy absorption at the point D, and the velocity of the cavity boundary near D is unbounded and directed into the side of the wedge face. If however $C < 0$, then there is energy production and an unbounded negative pressure acts at the edge of the wedge near D. Hence one must put $C = 0$ and drop the term $\varphi(w)$ in (2.5).

The constant υ_0 can be determined from the condition that $V(w)$ decreases fairly rapidly as $w \to \infty$. A series expansion of the right-hand side of (2.5) in powers of $1/w$ and equating the coefficient of $1/w$ to zero gives the following relation for υ_0

$$\upsilon_0 = \left[\alpha_2 \int_{u_0}^{1} \xi(t)dt - \alpha_1 \int_{-1}^{u_c} \xi(t)dt\right] \Big/ \int_{u_c}^{u_0} \xi(t)dt$$

The constant u_0 should be found from Eq. (2.3). The left-hand side of this equation, i.e. the function $F(u_0)$, has the property

$$F(1) \equiv 0 \tag{2.6}$$

This is because the functions $V(w)$, $z(w)$ are analytic in the half-plane $\operatorname{Im} w > 0$ and $V(w)$ decreases fairly rapidly at infinity, so that

$$\int_{-\infty}^{\infty} V(u) z'(u) du \equiv 0$$

Hence, (2.6) follows from this and from conditions (2.1).

3. We will consider the conditions for the various forms of fluid motion to exist. The type of motion with a cavity is called regime 1, and the type of motion where the left face of the wedge is unwetted is called regime 2.

Non-separated motion obviously corresponds to the case $u_0 = u_C$. Putting $u_0 = u_C$ in (2.5) and requiring that the complex velocity $V(w)$ decreases more rapidly than $1/w$ at infinity, we obtain the relation

$$\gamma = \alpha_2 / \alpha_1 = \gamma_1(\delta), \quad \gamma_1(\delta) = \int_{-1}^{u_c} \left(\frac{1-u}{1+u}\right)^{\delta} du \Big/ \int_{u_c}^{1} \left(\frac{1-u}{1+u}\right)^{\delta} du \tag{3.1}$$

which is the condition for non-separated fluid motion. Thus non-separated motion can only occur when there is a definite relation between the angles β, α_1, α_2.

The case $u_0 = 1$ corresponds to regime 2. Using relation (2.6) we conclude that $u_0 = 1$ when $p_0 > 0$ is not a root of Eq. (2.3), regime 2 is impossible, and both faces of the wedge are wetted by the fluid for all values of the parameters, even in the case of a plate when $\alpha = 0$. It can be shown that for $p_0 > 0$ Eq. (2.3) has a root $u_0 < 1$ for all $\delta \in (-1/2,\ 1/2)$, $\gamma \in [-1,\ \gamma_1(\delta))$, i.e. only two types of motion exist: non-separated flow if relation (3.1) is satisfied, or regime 1.

If $p_0 = 0$, then $u_0 = 1$ is a root of Eq. (2.3) and regime 2 is possible. In a small neighbourhood of the point $u_0 = 1$, we can use (2.6) to represent (2.3) in the form

$$C(\delta, \alpha_1, \alpha_2) G(\delta) \varepsilon = O(\varepsilon^{\mu}) \tag{3.2}$$

$$\mu > 1, \varepsilon = (1-u_0)^{1/2+\delta}, \quad C(\delta, \alpha_1, \alpha_2) = \alpha_1 A(\delta) + \alpha_2 B(\delta)$$

$$A(\delta) = \int_{-1}^{-2\delta} q(u) du \int_{0}^{\infty} \left[(1+t^2)^{\delta} - t^{2\delta}\right] dt$$

$$B(\delta) = \int_{-2\delta}^{1} q(u) du \int_{0}^{1} (1-t^2)^{\delta} dt, \quad q(u) = \sqrt{|u+2\delta|}\ (1-u)^{-1/2+\delta} (1+u)^{-\delta}$$

The equality $C(\delta,\ \alpha_1,\ \alpha_2) = 0$ is a necessary condition for the existence of the root $u_0 < 1$ of Eq. (3.2). from which it follows that

$$\gamma = \alpha_2 / \alpha_1 = \gamma_2(\delta) = -A(\delta) / B(\delta) \tag{3.3}$$

Relation (3.3) connecting γ and δ is the condition for the transition from regime 1 to regime 2.

In Fig. 3 curves 1 and 2 were computed from formulae (3.1) and (3.3), respectively, for the case $p_0 = 0$ and plotted in the (δ, Γ) plane, where $\Gamma = \gamma/(1+\gamma) = \alpha_2/\alpha$. The points on curve 1 refer to non-separated fluid flow, the points in the plane lying between curves 1 and 2 correspond to regime 1, and the points lying below curve 2 correspond to regime 2. The dashed curves 3–6 connect points in the plane between $(-1/2, 1)$ and $(1/2, 0)$. These are the curves $l(\delta, \Gamma) = l_0 = \text{const}$ along which the size of the cavity is constant and equal to l_0. Curves 3–6 correspond to the values $l_0 = 0.01, 0.5, 0.99, 0.999$.

The velocity υ and pressure p have singularities at points A, C and B (Fig. 1). We denote by R the distances from these points. As $R \to 0$ we have: for non-separated flow at the apex of the wedge υ, $p \sim \ln R$, and for regimes 1 and 2 υ, $p \sim R^{-1/4}$. At points A and B we have $\upsilon \sim R^{\delta(1/2-\delta)}$, $\upsilon \sim R^{-\delta/(1/2+\delta)}$ respectively, and $p = 0$. At point D the velocity and pressure are continuous.

4. We introduce a new dimensionless velocity υ/α_1 and pressure p/α_1, and retain the previous notation. Then in the case of regime 1 the problem under consideration depends on the three parameters δ, γ, p_0, and in the other cases only on the single parameter δ.

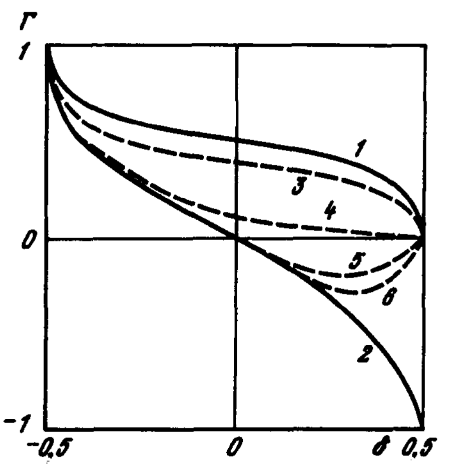


Fig. 3.

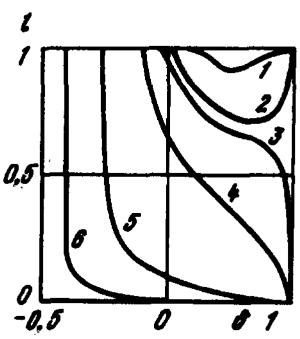
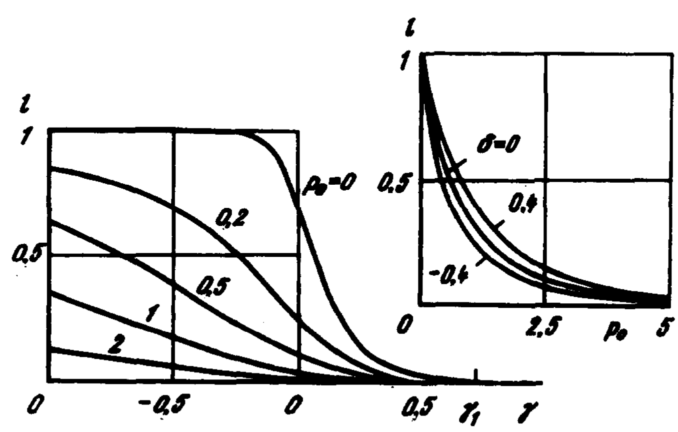


Fig. 4.



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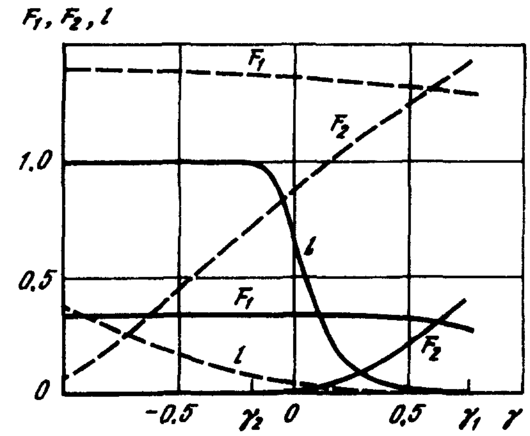

Fig. 6.

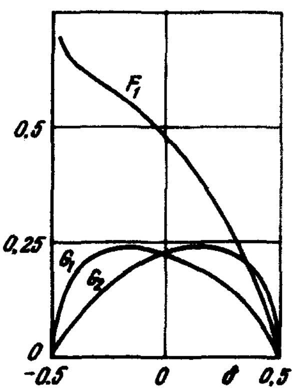

Fig. 7.

Figure 4 shows the results of calculations of $l(\delta)$ for $p_0 = 0$ and $\gamma = -0.1$, -0.01, 0.01, 0.1, 0.4, 1 (curves 1–6 respectively). When $\gamma > 0$ the function $l(\delta)$ decreases monotonically as δ increases and then, increasing, again reaches the value 1. For all $\gamma > 0$ there is always a value of δ for which the fluid motion is non-separated, whereas for all $\gamma < 0$ and δ the flow separates from a side of the wedge.

Figure 5 shows the dependence of the size of the cavity l formed at the left side of the plate on the pressure p_0 and the parameters δ and γ.

The curves on the left side of Fig. 5 are for $\delta = 0.25$.

Figure 6 shows the dependence of $l(\gamma)$, the force $F_1(\gamma)$ acting on the right face of the wedge, and the force $F_2(\gamma)$ acting on the left face of the wedge, on γ when $\delta = 0.2$. The continuous curves correspond to he pressure $p_0 = 0$, and the dashed lines to $-p_0 = 1$. When $\gamma = \gamma_1$ the motion is non-separated. If $-1 \leq \gamma \leq \gamma_2$, $p_0 = 0$, then the left face is not wetted by the fluid.

Figure 7 shows the forces $F_1(\delta)$ acting on the right side of the plate, the left side of which is not wetted by the fluid ($p_0 = 0$) and the forces $G_1(\delta)$, $G_2(\delta)$ in the case of non-separated motion. These forces G_1 and G_2 differ from F_1 and F_2 because they are normalized not on α_1, as was previously pointed out, but on $\alpha = \alpha_1 + \alpha_2$.

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